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Combinatorial Properties of the Kazhdan–Lusztig R-Polynomials for S_n

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We point out a deep and surprising connection between the Kazhdan–Lusztig R-polynomials for S_n and the enumeration and combinatorics of increasing subsequences in permutations. This leads to a simple combinatorial recurrence and to

1. INTRODUCTION

In their fundamental paper [18] Kazhdan and Lusztig defined, for every Coxeter group W, a family of polynomials, indexed by pairs of elements of W, which have become known as the Kazhdan-Lusztig polynomials of W (see, e.g., [16], Chap. 7). These polynomials are intimately related to the Bruhat order of W and to the algebraic geometry of Schubert varieties and have proven to be of fundamental importance in representation theory.

In order to prove the existence of these polynomials Kazhdan and Lusztig defined another family of polynomials (see [18], § 2) which are intimately related to the multiplicative structure of the Hecke algebra associated to W. These polynomials are now known as the R-polynomials of W (see, e.g., [16], § 7.5) and their importance stems mainly from the fact that their knowledge is equivalent to that of the Kazhdan–Lusztig polynomials.

Our aim in this work is to study some combinatorial properties of the R-polynomials of symmetric groups. More precisely, we point out a surprising connection between them and the enumeration and combinatorics of increasing subsequences in permutations. This leads to a simple combinatorial recurrence for computing these polynomials which in turn yields some new formulas for them. As a consequence of this recurrence, for example, we obtain a simple poset-theoretic condition on the interval determined by

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two elements in Bruhat order which insures that their R-polynomial is a power of $q-1$. This, in turn, implies an interesting connection between the Kazhdan–Lusztig polynomials of these intervals and generalized h-vectors.

The organization of the paper is as follows. In the next section we recall some basic definitions, notation, and results, both of an algebraic and combinatorial nature, that will be used in the rest of this work. In section 3 we prove our main result (Corollary 3.9). This gives a simple combinatorial recurrence, defined in terms of increasing subsequences of permutations, for computing the R-polynomials of symmetric groups. Finally, in section 4, we apply our main result to the explicit computation of Kazhdan–Lusztig and R-polynomials. More precisely, we derive a combinatorial formula for the R-polynomials (Theorem 4.1) and we single out some families of pairs of elements of S_n for which the corresponding R-polynomial has a simple closed form (Theorems 4.6 and 4.9). As a special case, we obtain the result that if the interval between two permutations in Bruhat order is a lattice then the corresponding R-polynomial is just a power of $q-1$ (Corollary 4.10), and the corresponding Kazhdan–Lusztig polynomial is the g-polynomial of the dual interval (Theorem 4.11). We also obtain sufficient conditions on two permutations so that their Kazhdan–Lusztig and R-polynomial factor (Theorem 4.4) and obtain as a consequence the result that Kazhdan–Lusztig and R-polynomials of symmetric groups are closed under products (Corollary 4.5).

The way in which our main result was discovered deserves a few words of comment. In October of 1991 Boris Shapiro attracted the author's attention to some polynomials arising from the computation of Euler characteristics of links of Schubert cells in the flags manifold (see [22]). These polynomials are naturally indexed by permutations and the main result of [22] is a combinatorial rule for computing them. Later Anders Björner suggested (and Boris Shapiro proved using topological arguments, see [23]) that these polynomials are the R-polynomials corresponding to *upper* intervals of the Bruhat order of S_n. Spurred by this I tried to generalize the combinatorial rule appearing in [22] and was thus led to the definition of the $\tilde{R}$-polynomials (see (19)) and then to the proof of the equivalence of these and the R-polynomials.

2. NOTATION AND PRELIMINARIES

In this section we collect some definitions, notation and results that will be used in the rest of this paper. We let $\mathbf{P} \stackrel{\text{def}}{=} \{1, 2, 3, ...\}$, $\mathbf{N} \stackrel{\text{def}}{=} \mathbf{P} \cup \{0\}$, and $\mathbf{Z}$ be the set of integers; for $a \in \mathbf{N}$ we let $[a] \stackrel{\text{def}}{=} \{1, 2, ..., a\}$ (where $[0] \stackrel{\text{def}}{=} \varnothing$). Given $n, m \in \mathbf{P}$, $n \leqslant m$, we let $[n, m] \stackrel{\text{def}}{=} [m] \backslash [n-1]$. We write

$S = \{a_1, ..., a_r\}_<$ to mean that $S = \{a_1, ..., a_r\}$ and $a_1 < \cdots < a_r$. The cardinality of a set A will be denoted by $|A|$. Given a polynomial $P(q)$, and $i \in \mathbf{N}$, we will denote by $[q^i](P(q))$ the coefficient of q^i in $P(q)$.

Given a set T we will let $S(T)$ be the set of all bijections $\pi: T \to T$, and $S_n \overset{\text{def}}{=} S([n])$. If $\sigma \in S(T)$ and $T \overset{\text{def}}{=} \{t_1, ..., t_r\}_< \subseteq \mathbf{P}$ then we write $\sigma = \sigma_1 \cdots \sigma_r$ to mean that $\sigma(t_i) = \sigma_i$, for $i = 1, ..., r$. If $\sigma \in S_n$ then we will also write σ in *disjoint cycle form* (see, e.g., [25], p. 17) and we will usually omit to write the 1-cycles of σ. For example, if $\sigma = 365492187$ then we also write $\sigma = (9, 7, 1, 3, 5)(2, 6)$. Given $\sigma, \tau \in S_n$ we let $\sigma\tau \overset{\text{def}}{=} \sigma \circ \tau$ (composition of functions) so that, for example, $(1, 2)(2, 3) = (1, 2, 3)$.

We will follow [25], Chap. 3, for notation and terminology concerning partially ordered sets. In particular, we say that a finite graded poset P with $\hat{0}$ and $\hat{1}$ is *Eulerian* if $\mu(x, y) = (-1)^{\rho(y) - \rho(x)}$ for all $x, y \in P$, $x \leqslant y$, where $\rho: P \to \mathbf{N}$ is the rank function of P. Recall (see, e.g., [25], § 3.14, p. 138, or [26], § 2, p. 190) that to any Eulerian poset P as above there are associated two polynomials, denoted $f(P; q)$ and $g(P; q)$, defined inductively as follows:

(i) if $|P| = 1$ then $f(P; q) \overset{\text{def}}{=} g(P; q) \overset{\text{def}}{=} 1$;

(ii) if P has rank $n+1 \geqslant 1$ and $f(P; q) = \sum_{i \geqslant 0} k_i q^i$ then

$$g(P; q) \overset{\text{def}}{=} \sum_{i=0}^{\lfloor n/2 \rfloor} (k_i - k_{i-1})\, q^i, \tag{1}$$

(where $k_{-1} \overset{\text{def}}{=} 0$);

(iii) if P has rank $n+1 \geqslant 1$ then

$$f(P; q) \overset{\text{def}}{=} \sum_{a \in P \setminus \{\hat{1}\}} g([\hat{0}, a]; q)(q-1)^{n - \rho(a)}. \tag{2}$$

The polynomials $f(P; q)$ and $g(P; q)$ were introduced in [26] and are two very subtle invariants of the Eulerian poset P (see [25], § 3.14 and [26], §§ 2, 3, for further information). We call $g(P; q)$ the *g-polynomial* of P, and $(h_0, ..., h_n)$, where $h_i \overset{\text{def}}{=} [q^{n-i}](f(P; q))$, for $i = 0, ..., n$, the *h-vector* of P.

We will follow [16] for general Coxeter groups notation and terminology. Given a Coxeter system (W, S) and $\sigma \in W$ we denote by $l(\sigma)$ the length of σ in W, with respect to S, and we let

$$D(\sigma) \overset{\text{def}}{=} \{s \in S: l(\sigma s) < l(\sigma)\},$$

and

$$d(\sigma) = |D(\sigma)|.$$

We call $D(\sigma)$ the *descent set* of σ and say that σ has $d(\sigma)$ *descents*. We denote by e the identity of W, and we let $T \stackrel{\text{def}}{=} \{\sigma s \sigma^{-1} : \sigma \in W, s \in S\}$. If $A \subseteq W$ and $x \in W$ we let $Ax \stackrel{\text{def}}{=} \{wx : w \in A\}$. We will always assume that W is partially ordered by (strong) *Bruhat order*. Recall (see, e.g., [16], § 5.9) that this means that $x \leqslant y$ if and only if there exist $r \in \mathbf{N}$ and $t_1, ..., t_r \in T$ such that $t_r \cdots t_1 x = y$ and $l(t_i \cdots t_1 x) > l(t_{i-1} \cdots t_1 x)$ for $i = 1, ..., r$. For example, the Hasse diagram of the Bruhat order on S_3 is shown in Figure 1. It is well known (see, e.g., [3], Corollary 1) that intervals of W (and their duals) are Eulerian posets.

We denote by $\mathscr{H}(W)$ the *Hecke algebra* associated to W. Recall (see, e.g., [16], Chap. 7) that this is the free $\mathbf{Z}[q, q^{-1}]$-module having the set $\{T_w : w \in W\}$ as a basis and multiplication such that

$$T_w T_s = \begin{cases} T_{ws}, & \text{if} \quad l(ws) > l(w), \\ qT_{ws} + (q-1)\, T_w, & \text{if} \quad l(ws) < l(w), \end{cases} \tag{3}$$

for all $w \in W$ and $s \in S$. It is well known that this is an associative algebra having T_e as unity and that each basis element is invertible in $\mathscr{H}(W)$. More precisely, we have the following result (see, [16], Proposition 7.4).

PROPOSITION 2.1. *Let $y \in W$. Then*

$$(T_{y^{-1}})^{-1} = q^{-l(y)} \sum_{x \leqslant y} (-1)^{l(y)-l(x)}\, R_{x,y}(q)\, T_x,$$

where $R_{x,y}(q) \in \mathbf{Z}[q]$.

The polynomials $R_{x,y}$ defined by the previous proposition are called the *R-polynomials* of W. It is easy to see that $\deg(R_{x,y}) = l(y) - l(x)$, and that $R_{x,x}(q) = 1$, for all $x, y \in W$, $x \leqslant y$. It is customary to let $R_{x,y}(q) \stackrel{\text{def}}{=} 0$ if

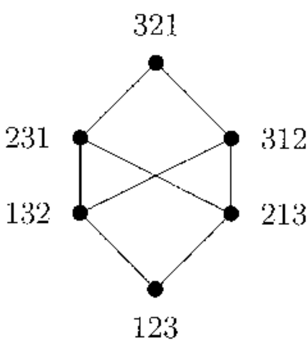

FIG. 1. The Bruhat order on S_3.

$x \nleqslant y$. We then have the following fundamental result that follows from (3) and Proposition 2.1 (see [16], § 7.5).

THEOREM 2.2. *Let $x, y \in W$ and $s \in D(y)$. Then*

$$R_{x,y}(q) = \begin{cases} R_{xs,ys}(q), & \text{if } s \in D(x), \\ qR_{xs,ys}(q) + (q-1)\, R_{x,ys}, & \text{if } s \notin D(x). \end{cases} \tag{4}$$

Note that the preceding theorem can be used to inductively compute the R-polynomials since $l(ys) < l(y)$. Therefore, one could take Theorem 2.2 as the definition of the R-polynomials, together with the initial conditions that $R_{x,x}(q) = 1$ and $R_{x,y}(q) = 0$, for all $x, y \in W$, $x \nleqslant y$.

Even though our interest in this work is mainly in the R-polynomials, some of our results have consequences also for the Kazhdan–Lusztig polynomials, which we now define. The following result is not hard to prove (and, in fact, holds in much greater generality, see [27], Corollary 6.7 and Example 6.9) and a proof can be found, e.g., in [16], §§ 7.9–11, or [18], § 2.2.

THEOREM 2.3. *There is a unique family of polynomials $\{P_{x,y}(q)\}_{x,y \in W} \subseteq$ $\mathbf{Z}[q]$, such that, for all $x, y \in W$:*

(i) $P_{x,y}(q) = 0$ *if* $x \nleqslant y$;

(ii) $P_{x,x}(q) = 1$;

(iii) $\deg(P_{x,y}(q)) \leqslant \lfloor (l(y) - l(x) - 1)/2 \rfloor$, *if* $x \leqslant y$;

(iv) $$q^{l(y)-l(x)} P_{x,y}\left(\frac{1}{q}\right) = \sum_{x \leqslant z \leqslant y} R_{x,z}(q)\, P_{z,y}(q), \tag{5}$$

if $x \leqslant y$.

The polynomials $P_{x,y}(q)$ defined by the preceding theorem are called the *Kazhdan–Lusztig* polynomials of W. Note that parts (iii) and (iv) of Theorem 2.3 actually yield an inductive procedure to compute the polynomials $P_{x,y}(q)$ for all $x, y \in W$, taking parts (i) and (ii) as initial conditions. The polynomials $P_{x,y}(q)$ have been the subject of considerable study, and we refer the reader to, e.g., [16], Chapter 7, for further information about them.

The object of this work is to study some combinatorial properties of the R-polynomials for the symmetric groups. Therefore, from now on we will let $W = S_n$ and $S = \{s_1, ..., s_{n-1}\}$ where $s_i \stackrel{\text{def}}{=} (i, i+1)$, for $i = 1, ..., n-1$, and we will say that $p(q) \in \mathbf{Z}[q]$ is an *R-polynomial* (respectively, a *Kazhdan–Lusztig polynomial*) if there are $n \in \mathbf{P}$ and $\sigma, \tau \in S_n$ such that $p(q) = R_{\sigma,\tau}(q)$ (respectively, $P_{\sigma,\tau}(q)$). We will often find it convenient to identify $S(T)$

with $S_{|T|}$ for $T \subseteq \mathbf{P}$. We will do this by identifying t_i with i, for $i=1, ..., r$, if $\{t_1, ..., t_r\}_< \stackrel{\text{def}}{=} T$. So, for example, $R_{5294, 4592}(q) \stackrel{\text{def}}{=} R_{3142, 2341}(q)$.

It is clear from (3), Proposition 2.1, and Theorem 2.2 that the R-polynomials depend heavily on the Bruhat order, the length function, and descent sets. Therefore, we will find it useful to have combinatorial descriptions of these objects for S_n. The following result is well known and a proof can be found, e.g., in [21], Chapter 1.

PROPOSITION 2.4. *Let $\sigma \in S_n$, and $i \in [n-1]$. Then:*

(i) $l(\sigma) = |\{(a, b) \in [n] \times [n] : a < b, \sigma(a) > \sigma(b)\}|$;

(ii) $s_i \in D(\sigma)$ *if and only if* $\sigma(i) > \sigma(i+1)$.

For example, if $\sigma = 615243$ then $l(\sigma) = 9$ and $D(\sigma) = \{(1, 2), (3, 4), (5, 6)\}$. The following characterization of the Bruhat order of S_n is also well known, and will be used repeatedly in this work. We refer the reader to, e.g., [21], Chapter 1, for a proof. For $\sigma \in S_n$, and $i \in [n]$, let

$$\{\sigma^{i,1}, ..., \sigma^{i,i}\}_< \stackrel{\text{def}}{=} \{\sigma(1), ..., \sigma(i)\}. \tag{6}$$

THEOREM 2.5. *Let $\sigma, \tau \in S_n$. Then $\sigma \leqslant \tau$ if and only if $\sigma^{i,j} \leqslant \tau^{i,j}$ for all $1 \leqslant j \leqslant i \leqslant n-1$.*

For example, if $\sigma = 4123$ and $\tau = 2431$ then $(\sigma^{1,1}, \sigma^{2,1}, \sigma^{2,2}, ..., \sigma^{3,3}) = (4, 1, 4, 1, 2, 4)$ and $(\tau^{1,1}, \tau^{2,1}, \tau^{2,2}, ..., \tau^{3,3}) = (2, 2, 4, 2, 3, 4)$ and hence σ and τ are incomparable in Bruhat order.

3. R-POLYNOMIALS AND INCREASING SUBSEQUENCES

In this section we derive our main result. This gives a simple combinatorial recurrence for the R-polynomials in terms of increasing subsequences of permutations.

We start by studying some basic combinatorics of increasing subsequences in a permutation. These results will be useful later on in our study of the R-polynomials. Let $\sigma \in S_n$, and $s \in S$. For $a, b, i, j \in [n]$ we let

$$\mathscr{C}_{i,j}(\sigma) \stackrel{\text{def}}{=} \{(\sigma(i_k), ..., \sigma(i_1)) \in S_n : k \in [n], i = i_1 < i_2 < \cdots < i_k = j, \\ \sigma(i_1) < \sigma(i_2) < \cdots < \sigma(i_k)\},$$

$$\mathscr{C}_{i,j}(\sigma; s) \stackrel{\text{def}}{=} \{w \in \mathscr{C}_{i,j}(\sigma) : s \in D(w\sigma)\},$$

$$\mathscr{C}(\sigma) \stackrel{\text{def}}{=} \bigcup_{1 \leqslant i < j \leqslant n} \mathscr{C}_{i,j}(\sigma),$$

and

$$T_\sigma[i, j; a, b] \overset{\text{def}}{=} \{r \in [n] : i < r < j, a < \sigma(r) < b\}.$$

Given $w = (\sigma(i_k), ..., \sigma(i_1)) \in \mathscr{C}_{i, j}(\sigma)$ we let

$$n(w, \sigma) \overset{\text{def}}{=} \sum_{r=1}^{k-1} |T_\sigma[i_r, i_{r+1}; \sigma(i_r), \sigma(j)]|, \tag{7}$$

$$m(w, \sigma) \overset{\text{def}}{=} \sum_{r=1}^{k-1} |T_\sigma[i_r, i_{r+1}; \sigma(i_{r+1}), \sigma(j)]|, \tag{8}$$

and

$$p(w, \sigma) \overset{\text{def}}{=} \sum_{r=1}^{k-1} (k-1-r)\, |T_\sigma[i_r, i_{r+1}; \sigma(i_{r+1}), \sigma(j)]|. \tag{9}$$

We also let $k(\sigma)$ be the length of the longest cycle of σ, and $F(\sigma)$ be the set of fixed points of σ. For example, if $\sigma = 215496378$ then $k(\sigma) = 5$, $F(\sigma) = \{4, 6\}$, $T_\sigma[2, 6; 4, 8] = \{3\}$, $\mathscr{C}_{1, 6}(\sigma) = \{(6, 2), (6, 5, 2), (6, 4, 2)\}$, and if $w = (8, 6, 4, 2) \in \mathscr{C}_{1, 9}(\sigma)$ then $n(w, \sigma) = 1 + 0 + 1 = 2$, $m(w, \sigma) = 1 + 0 + 0 = 1$, and $p(w, \sigma) = 2 \cdot 1 + 1 \cdot 0 + 0 \cdot 0 = 2$. Note that $\mathscr{C}_{i, j}(\sigma) \neq \varnothing$ if and only if $i \leqslant j$ and $\sigma(i) \leqslant \sigma(j)$.

We begin by investigating some properties of the sets $\mathscr{C}_{i, j}(\sigma)$.

Proposition 3.1. *Let $\sigma \in S_n$, and $w \in \mathscr{C}(\sigma)$. Then*:

(i) $\sigma < w\sigma$;

(ii) $l(w\sigma) - l(\sigma) = 2n(w, \sigma) + k(w) - 1$.

Proof. Let $w \overset{\text{def}}{=} (\sigma(i_k), ..., \sigma(i_1)) \in \mathscr{C}(\sigma)$ and let, for convenience, $t_r \overset{\text{def}}{=} (\sigma(i_k), \sigma(i_r))$ for $r = 1, ..., k-1$. Then, clearly,

$$w\sigma = t_1 t_2 \cdots t_{k-1} \sigma \tag{10}$$

and since $\sigma(i_k) > \sigma(i_r)$ and $i_r < i_k$, for $r = 1, ..., k-1$, we have that

$$l(t_r t_{r+1} \cdots t_{k-1} \sigma) - l(t_{r+1} \cdots t_{k-1} \sigma) = 2\, |T_\sigma[i_r, i_{r+1}; \sigma(i_r), \sigma(i_k)]| + 1, \tag{11}$$

for $r = 1, ..., k-1$, and (i) follows. Summing (11) for $r = 1, ..., k-1$ and using (7) and (10) yields (ii), as desired. ∎

A little more involved is the next property, which will be used in the proof of Theorem 4.9.

PROPOSITION 3.2. *Let $\sigma \in S_n$, and $i, j \in [n]$ be such that $\mathscr{C}_{i,j}(\sigma) \neq \emptyset$. Then there exists a unique $w \in \mathscr{C}_{i,j}(\sigma)$ such that $n(w, \sigma) = 0$.*

Proof. We may assume that $i < j$ and $\sigma(i) < \sigma(j)$. Now let $i_1 \stackrel{\text{def}}{=} i$ and

$$i_{r+1} \stackrel{\text{def}}{=} \begin{cases} \min(T_\sigma[i_r, j+1; \sigma(i_r), \sigma(j)+1]), & \text{if} \quad i_r < j \text{ and } \sigma(i_r) < \sigma(j), \\ j, & \text{otherwise}. \end{cases} \tag{12}$$

for $r = 1, 2, ...$ Then, clearly

$$i = i_1 < i_2 < \cdots < i_k = j, \tag{13}$$

and

$$\sigma(i) = \sigma(i_1) < \sigma(i_2) < \cdots < \sigma(i_k) = \sigma(j) \tag{14}$$

where $k \stackrel{\text{def}}{=} \min\{r \in \mathbf{P} : i_r = j\}$. Therefore $w_0 \stackrel{\text{def}}{=} (\sigma(i_k), ..., \sigma(i_1)) \in \mathscr{C}_{i,j}(\sigma)$. Furthermore, for any $r \in [k-1]$ we have that

$$T_\sigma[i_r, i_{r+1}; \sigma(i_r), \sigma(j)] = \emptyset,$$

(for if $s \in T_\sigma[i_r, i_{r+1} : \sigma(i_r), \sigma(j)]$ then $s < i_{r+1}$ and $s \in T_\sigma[i_r, j+1; \sigma(i_r), \sigma(j)+1]$, which contradicts (12)). Hence, by (7), $n(w_0, \sigma) = 0$.

Now let $w \stackrel{\text{def}}{=} (\sigma(j_m), ..., \sigma(j_1))$ be such that $w \in \mathscr{C}_{i,j}(\sigma)$ and $n(w, \sigma) = 0$. Then

$$i = j_1 < j_2 < \cdots < j_m = j, \tag{15}$$

$$\sigma(i) = \sigma(j_1) < \sigma(j_2) < \cdots < \sigma(j_m) = \sigma(j), \tag{16}$$

and, by (7),

$$T_\sigma[j_r, j_{r+1}; \sigma(j_r), \sigma(j)] = \emptyset \tag{17}$$

for $r = 1, ..., m-1$. We now claim that $i_r = j_r$ for $r = 1, ..., \min(k, m)$. We will prove this by induction on r, the claim being clearly true if $r = 1$. So let $t \in [\min(k, m) - 1]$ and assume that $i_r = j_r$ for $r = 1, ..., t$. Then from (15) and (16) there follows that $j_{t+1} \in T_\sigma[i_t, j+1; \sigma(i_t), \sigma(j)+1]$. This, by (12), (13), and (14), implies that

$$i_{t+1} \leqslant j_{t+1}.$$

Now, if $i_{t+1} < j_{t+1}$, then by (13), (14), and our induction hypothesis we have that

$$i_{t+1} \in T_\sigma[j_t, j_{t+1}; \sigma(j_t), \sigma(j)],$$

which contradicts (17). Therefore $i_{t+1} = j_{t+1}$ and this concludes the induction step and proves our claim. In particular, this shows that $i_{\min(k,m)} = j_{\min(k,m)} = j$ and hence, by (13) and (15), that $k = m$. Hence $w = w_0$ and this completes the proof. ∎

We now define a distance on S_n which will play a crucial role in all that follows. For $\sigma, \tau \in S_n$ we let

$$d(\sigma, \tau) \stackrel{\text{def}}{=} \max\{i \in [n] : \sigma^{-1}(i) \neq \tau^{-1}(i)\}, \tag{18}$$

(where $\max\{\varnothing\} \stackrel{\text{def}}{=} 0$). So, for example, $d(198265374, 298461357) = \max\{1, 2, 5, 7, 4\} = 7$, and $d(\sigma, \tau) \neq 1$ for all $\sigma, \tau \in S_n$.

The next result shows that the function $d: S_n \times S_n \to \mathbf{N}$ just defined is indeed a distance on S_n, and that it is invariant under right translations.

Proposition 3.3. *For all $\sigma, \tau, w \in S_n$ we have that:*

(i) $d(\sigma, \tau) = 0$ *if and only if* $\sigma = \tau$;

(ii) $d(\sigma, \tau) = d(\tau, \sigma)$;

(iii) $d(\sigma, \tau) \leqslant d(\sigma, w) + d(w, \tau)$;

(iv) $d(\sigma w, \tau w) = d(\sigma, \tau)$.

Proof. (i), (ii), and (iv) are clear. To prove (iii) let $j \stackrel{\text{def}}{=} d(\sigma, \tau)$ and assume $j > 0$. Then $\sigma^{-1}(j) \neq \tau^{-1}(j)$ and hence either $\sigma^{-1}(j) \neq w^{-1}(j)$ or $w^{-1}(j) \neq \tau^{-1}(j)$. Therefore either $d(\sigma, w) \geqslant j$ or $d(w, \tau) \geqslant j$ and hence

$$d(\sigma, \tau) \leqslant \max\{d(\sigma, w), d(w, \tau)\},$$

and (iii) follows. ∎

The next result gives what are for us the crucial properties of the distance function.

Proposition 3.4. *Let $\sigma, \tau \in S_n$ be such that $\sigma < \tau$. Then:*

(i) $\tau^{-1}(d(\sigma, \tau)) < \sigma^{-1}(d(\sigma, \tau))$;

(ii) $\sigma(\tau^{-1}(d(\sigma, \tau))) < d(\sigma, \tau)$.

Proof. By the definition of the Bruhat order it is clearly enough to prove (i) when $\tau = (\sigma(i), \sigma(j))\sigma$ with $i < j$ and $\sigma(i) < \sigma(j)$. In this case

$$d(\sigma, \tau) = \max\{\sigma(i), \sigma(j)\} = \sigma(j),$$

and therefore

$$\tau^{-1}(d(\sigma,\tau))=\tau^{-1}(\sigma(j))=\sigma^{-1}(\sigma(i))=i<j=\sigma^{-1}(\sigma(j))=\sigma^{-1}(d(\sigma,\tau)),$$

as desired.

To prove (ii) let $k \stackrel{\text{def}}{=} \sigma(\tau^{-1}(d(\sigma,\tau)))$. Then $\sigma^{-1}(k)=\tau^{-1}(d(\sigma,\tau))$ and hence, by (18), $k \neq d(\sigma,\tau)$. But if $k>d(\sigma,\tau)$ then, by (18), $\sigma^{-1}(k)=\tau^{-1}(k)$ and hence $\tau^{-1}(k)=\tau^{-1}(d(\sigma,\tau))$ which is impossible since $k \neq d(\sigma,\tau)$. Therefore $k<d(\sigma,\tau)$, as desired. ∎

We are now ready to define, in terms of increasing subsequences, some polynomials which will turn out to be, essentially, the R-polynomials for S_n. For $\sigma, \tau \in S_n$ we define a polynomial $\tilde{R}_{\sigma,\tau}(t)$ by the following recurrence:

$$\tilde{R}_{\sigma,\tau}(t) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } \sigma \not\leqslant \tau, \\ \displaystyle\sum_{w \in \mathscr{C}_{\tau^{-1}(d),\,\sigma^{-1}(d)}(\sigma)} t^{k(w)-1}\tilde{R}_{w\sigma,\tau}(t), & \text{if } \sigma<\tau, \\ 1, & \text{if } \sigma=\tau, \end{cases} \tag{19}$$

where $d \stackrel{\text{def}}{=} d(\sigma,\tau)$. Note that, by Proposition 3.4, $\mathscr{C}_{\tau^{-1}(d),\,\sigma^{-1}(d)}(\sigma) \neq \varnothing$ if $\sigma<\tau$, and, by Proposition 3.1, $l(\tau)-l(w\sigma)<l(\tau)-l(\sigma)$ for all $w \in \mathscr{C}_{\tau^{-1}(d),\,\sigma^{-1}(d)}(\sigma)$, so that (19) inductively defines the polynomials $\tilde{R}_{\sigma,\tau}(t)$ for all $\sigma, \tau \in S_n$.

We illustrate the definition with an example. Let $\sigma=1342$ and $\tau=4321$, then applying (19) repeatedly we have that

$$\begin{aligned}\tilde{R}_{1342,\,4321}(t) &= t\tilde{R}_{4312,\,4321}(t)+t^2\tilde{R}_{4132,\,4321}(t)\\ &= t^2\tilde{R}_{4321,\,4321}(t)+t^3\tilde{R}_{4312,\,4321}(t)\\ &= t^2+t^4\tilde{R}_{4321,\,4321}(t)\\ &= t^2+t^4.\end{aligned}$$

It is a general fact that the polynomials $\tilde{R}_{\sigma,\tau}(t)$ are, essentially, polynomials in t^2. More precisely, we have the following result.

PROPOSITION 3.5. *For $\sigma, \tau \in S_n, \sigma \leqslant \tau$, there exists a unique polynomial $Q_{\sigma,\tau}(t)$ such that*

$$\tilde{R}_{\sigma,\tau}(t)=\begin{cases} Q_{\sigma,\tau}(t^2), & \textit{if } \ l(\tau)-l(\sigma) \textit{ is even}, \\ tQ_{\sigma,\tau}(t^2), & \textit{if } \ l(\tau)-l(\sigma) \textit{ is odd}. \end{cases}$$

Proof. We proceed by induction on $d(\sigma,\tau)$, the thesis being clear if $\sigma=\tau$. So let $\sigma<\tau$, $d \stackrel{\text{def}}{=} d(\sigma,\tau)$, and suppose that $l(\tau)-l(\sigma)$ is even. Then, by Proposition 3.1, we have that

$$l(\tau)-l(w\sigma) \equiv l(\tau)-l(\sigma)-k(w)+1 \equiv -k(w)+1 (\mathrm{mod}\ 2)$$

for any $w \in \mathscr{C}_{i,j}(\sigma)$ (where $i \stackrel{\text{def}}{=} \tau^{-1}(d)$, $j \stackrel{\text{def}}{=} \sigma^{-1}(d)$). Therefore we conclude from (19) and our induction hypotheses that

$$\tilde{R}_{\sigma,\tau}(t) = \sum_{\{w \in \mathscr{C}_{i,j}(\sigma) : k(w) \equiv 1(2)\}} t^{k(w)-1} Q_{w\sigma,\tau}(t^2) + \sum_{\{w \in \mathscr{C}_{i,j}(\sigma) : k(w) \equiv 0(2)\}} t^{k(w)} Q_{w\sigma,\tau}(t^2),$$

which is clearly a polynomial in t^2. A similar argument is used if $l(\tau) - l(\sigma)$ is odd. ▮

We now wish to show that the polynomials $\tilde{R}_{\sigma,\tau}(t)$ are, up to a simple transformation, the *R*-polynomials for S_n. We will do this by showing that they satisfy an appropriate version of the basic recurrence (4). In order to do this, in view of our definition (19), it is necessary to investigate what happens to the sets $\mathscr{C}_{i,j}(\sigma)$ when an adjacent transposition is applied to σ. This requires a delicate combinatorial analysis whose result is given in the following lemma.

LEMMA 3.6. *Let $\pi \in S_n$, $a, b, c \in [n]$ be such that $a < b$, $\pi(a) < \pi(b)$, and $\pi(c) < \pi(c+1)$, and $s \stackrel{\text{def}}{=} (c, c+1)$, $\bar{s} \stackrel{\text{def}}{=} (\pi(c), \pi(c+1))$. Then*

$$\mathscr{C}_{s(a),s(b)}(\pi s) = \begin{cases} \mathscr{C}_{a,b}(\pi) \backslash \mathscr{C}_{a,b}(\pi; s)\bar{s}, & \\ \qquad \text{if } \ a < c < b, \text{ and } \pi(c), \pi(c+1) \in [\pi(a), \pi(b)], & \\ \mathscr{C}_{a,b}(\pi) \backslash \mathscr{C}_{a+1,b}(\pi)\, \bar{s}, & \\ \qquad \text{if } \ a = c, \text{ and } \pi(c+1) \in [\pi(a), \pi(b)], & \\ \mathscr{C}_{a,b}(\pi), & \text{otherwise.} \end{cases}$$

Proof. Suppose first that $a < c < b$ and $\pi(a) < \pi(c) < \pi(c+1) \leqslant \pi(b)$. It is then clear that $\mathscr{C}_{s(a),s(b)}(\pi s) \subseteq \mathscr{C}_{a,b}(\pi)$ and that if $w \in \mathscr{C}_{a,b}(\pi) \backslash \mathscr{C}_{s(a),s(b)}(\pi s)$ then there exist $p \in \mathbf{P}$, $q \in \mathbf{N}$ and indices $i_1, ..., i_p, j_0, ..., j_q \in [n]$ such that

$$w = (\pi(j_q), ..., \pi(j_1), \pi(c+1), \pi(c), \pi(i_p), ..., \pi(i_1)),$$

$$a = i_1 < \cdots < i_p < c < c+1 = j_0 < j_1 < \cdots < j_q = b,$$

and

$$\pi(i_1) < \cdots < \pi(i_p) < \pi(c) < \pi(c+1) < \pi(j_1) < \cdots < \pi(j_q).$$

Therefore

$$w\bar{s} = (\pi(j_q), ..., \pi(j_1), \pi(c+1), \pi(i_p), ..., \pi(i_1)),$$

and hence $w\bar{s} \in \mathscr{C}_{a,b}(\pi)$. Furthermore, $(w\bar{s}\pi)(c) = \pi(c) > \pi(i_p) = (w\bar{s}\pi)(c+1)$, so that $s \in D(w\bar{s}\pi)$.

Conversely, if $u \in \mathscr{C}_{a,b}(\pi)$ and $s \in D(u\pi)$ then $\pi(c+1) \notin F(u)$ and $\pi(c) \in F(u)$ (for if $\pi(c+1) \in F(u)$ then $s \notin D(u\pi)$). Hence there exist $p \in \mathbf{P}$, $q \in \mathbf{N}$ and indices $i_1, ..., i_p, j_0, ..., j_q \in [n]$ such that

$$u = (\pi(j_q), ..., \pi(j_1), \pi(c+1), \pi(i_p), ..., \pi(i_1)),$$

$$a = i_1 < \cdots < i_p < c < c+1 = j_0 < j_1 < \cdots < j_q = b,$$

and

$$\pi(i_1) < \cdots < \pi(i_p) < \pi(c+1) < \pi(j_1) < \cdots < \pi(j_q).$$

Furthermore, $\pi(i_p) < \pi(c)$ (since $s \in D(u\pi)$). Therefore

$$u\bar{s} = (\pi(j_q), ..., \pi(j_1), \pi(c+1), \pi(c), \pi(i_p), ..., \pi(i_1))$$

and hence $u\bar{s} \in \mathscr{C}_{a,b}(\pi) \backslash \mathscr{C}_{s(a),s(b)}(\pi s)$.

Suppose now that $a = c$ and $\pi(a) < \pi(a+1) < \pi(b)$. Then it is clear that $\mathscr{C}_{s(a),s(b)}(\pi s) = \mathscr{C}_{a+1,b}(\pi s) \subseteq \mathscr{C}_{a,b}(\pi)$ and that, given $w \in \mathscr{C}_{a,b}(\pi)$, $w \notin \mathscr{C}_{a+1,b}(\pi s)$ if and only if $\pi(a+1) \notin F(w)$. But it is easy to see that

$$\{w \in \mathscr{C}_{a,b}(\pi) : \pi(a+1) \notin F(w)\}\ \bar{s} = \mathscr{C}_{a+1,b}(\pi).$$

and the result follows.

Finally, if either $c < a-1$, or $b < c$, or $\pi(c) \notin [\pi(a), \pi(b)]$, or $\pi(c+1) \notin [\pi(a), \pi(b)]$ (note that this covers all the possible remaining cases since if $c = b$ then, by our hypotheses, $\pi(c+1) > \pi(c) = \pi(b)$ and hence $\pi(c+1) \notin [\pi(a), \pi(b)]$, while if $c = a-1$ then, similarly, $\pi(c) < \pi(c+1) = \pi(a)$ and hence $\pi(c) \notin [\pi(a), \pi(b)]$) then it follows immediately from our definitions that $\mathscr{C}_{s(a),s(b)}(\pi s) = \mathscr{C}_{a,b}(\pi)$, and this concludes the proof. ∎

We can now prove one of the main results of this section.

THEOREM 3.7. *Let $\sigma, \tau \in S_n$ be such that $\sigma \leqslant \tau$. Then, for each $s \in D(\tau)$, we have that*

$$\tilde{R}_{\sigma,\tau}(t) = \begin{cases} \tilde{R}_{\sigma s, \tau s}(t), & \text{if } s \in D(\sigma), \\ \tilde{R}_{\sigma s, \tau s}(t) + t\tilde{R}_{\sigma, \tau s}(t), & \text{if } s \notin D(\sigma). \end{cases}$$

Proof. We proceed by induction on $d(\sigma, \tau)$, the thesis being clearly true if $d(\sigma, \tau) = 0$. So let $\sigma, \tau \in S_n$ be such that $d(\sigma, \tau) > 0$ and $\sigma < \tau$. Fix $s \in D(\tau)$

and let, for convenience, $s=(k,k+1)$, $d \stackrel{\text{def}}{=} d(\sigma,\tau)$ and $\bar{s} \stackrel{\text{def}}{=} (\sigma(k),\sigma(k+1))$ (so that $\bar{s}=\sigma s\sigma^{-1}$). Applying (19) we get that

$$\tilde{R}_{\sigma,\tau}(t)=\sum_{w\in\mathscr{C}_{i,j}(\sigma)} t^{k(w)-1}\tilde{R}_{w\sigma,\tau}(t), \tag{20}$$

where $i \stackrel{\text{def}}{=} \tau^{-1}(d)$, and $j \stackrel{\text{def}}{=} \sigma^{-1}(d)$. Also, since $d(\sigma s,\tau s)=d(\sigma,\tau)$, we get from (19) that

$$\tilde{R}_{\sigma s,\tau s}(t)=\sum_{w\in\mathscr{C}_{s(i),s(j)}(\sigma s)} t^{k(w)-1}\tilde{R}_{w\sigma s,\tau s}(t). \tag{21}$$

Finally, note that, if $s\notin D(\sigma)$ then $\sigma(k),\sigma(k+1)\leqslant d(\sigma,\tau)$ (for if $\sigma(k+1)>d$ then $\tau(k)>\tau(k+1)=\sigma(k+1)>d$ and hence $\sigma(k)=\tau(k)>\sigma(k+1)$, which is a contradiction) and hence $d(\sigma,\tau s)=d(\sigma,\tau)$, unless $k=i=j-1$. Therefore we conclude from (19) that

$$\tilde{R}_{\sigma,\tau s}(t)=\sum_{w\in\mathscr{C}_{s(i),j}(\sigma)} t^{k(w)-1}\tilde{R}_{w\sigma,\tau s}(t), \tag{22}$$

whenever $s\notin D(\sigma)$. Now, from (20) and our induction hypotheses (which can be applied since $d(w\sigma,\tau)<d(\sigma,\tau)$ for all $w\in\mathscr{C}_{i,j}(\sigma)$) we get that

$$\begin{aligned}\tilde{R}_{\sigma,\tau}(t)&=\sum_{w\in\mathscr{C}_{i,j}(\sigma;s)} t^{k(w)-1}\tilde{R}_{w\sigma s,\tau s}(t)\\&\quad+\sum_{\{w\in\mathscr{C}_{i,j}(\sigma):s\notin D(w\sigma)\}} t^{k(w)-1}(\tilde{R}_{w\sigma s,\tau s}(t)+t\tilde{R}_{w\sigma,\tau s}(t))\\&=\sum_{w\in\mathscr{C}_{i,j}(\sigma)} t^{k(w)-1}\tilde{R}_{w\sigma s,\tau s}(t)\\&\quad+\sum_{\{w\in\mathscr{C}_{i,j}(\sigma):s\notin D(w\sigma)\}} t^{k(w)}\tilde{R}_{w\sigma,\tau s}(t).\end{aligned} \tag{23}$$

There are now several cases to consider:

(i) $i+1\leqslant k\leqslant j$, and $\sigma(i)\leqslant\sigma(k+1)<\sigma(k)\leqslant\sigma(j)$.

Then applying Lemma 3.6 (with $\pi=\sigma s$, $a=s(i)$, $b=s(j)$, and $c=k$) yields that $\mathscr{C}_{s(i),s(j)}(\sigma s)=\mathscr{C}_{i,j}(\sigma)\uplus\mathscr{C}_{s(i),s(j)}(\sigma s;s)\bar{s}$, but it is easy to see that

$$\mathscr{C}_{s(i),s(j)}(\sigma s;s)=\{w\in\mathscr{C}_{i,j}(\sigma):s\notin D(w\sigma)\}.$$

Therefore, we conclude from (23) that

$$\begin{aligned}\tilde{R}_{\sigma,\tau}(t) &= \sum_{w\in\mathscr{C}_{i,j}(\sigma)} t^{k(w)-1}\tilde{R}_{w\sigma s,\tau s}(t) + \sum_{w\in\{w\in\mathscr{C}_{i,j}(\sigma):\, s\notin D(w\sigma)\}\bar{s}} t^{k(w\bar{s})}\tilde{R}_{w\bar{s}\sigma,\tau s}(t)\\ &= \sum_{w\in\mathscr{C}_{i,j}(\sigma)} t^{k(w)-1}\tilde{R}_{w\sigma s,\tau s}(t) + \sum_{w\in\{w\in\mathscr{C}_{i,j}(\sigma):\, s\notin D(w\sigma)\}\bar{s}} t^{k(w)-1}\tilde{R}_{w\sigma s,\tau s}(t)\\ &= \tilde{R}_{\sigma s,\tau s}(t), \end{aligned} \tag{24}$$

by (21).

(ii) $i+1\leqslant k\leqslant j$, and $\sigma(i)\leqslant\sigma(k)<\sigma(k+1)\leqslant\sigma(j)$.

Then by Lemma 3.6 we have that $\mathscr{C}_{s(i),s(j)}(\sigma s)=\mathscr{C}_{i,j}(\sigma)\backslash\mathscr{C}_{i,j}(\sigma;s)\,\bar{s}$. Therefore, we conclude from (23) that

$$\begin{aligned}\tilde{R}_{\sigma,\tau}(t) &= \sum_{w\in\mathscr{C}_{i,j}(\sigma)} t^{k(w)-1}(\tilde{R}_{w\sigma s,\tau s}(t)+t\tilde{R}_{w\sigma,\tau s}(t)) - \sum_{w\in\mathscr{C}_{i,j}(\sigma;s)} t^{k(w)}\tilde{R}_{w\sigma,\tau s}(t)\\ &= \sum_{w\in\mathscr{C}_{i,j}(\sigma)} t^{k(w)-1}(\tilde{R}_{w\sigma s,\tau s}(t)+t\tilde{R}_{w\sigma,\tau s}(t)) - \sum_{w\in\mathscr{C}_{i,j}(\sigma;s)\bar{s}} t^{k(w\bar{s})}\tilde{R}_{w\bar{s}\sigma,\tau s}(t)\\ &= \sum_{w\in\mathscr{C}_{i,j}(\sigma)} t^{k(w)-1}(\tilde{R}_{w\sigma s,\tau s}(t)+t\tilde{R}_{w\sigma,\tau s}(t)) - \sum_{w\in\mathscr{C}_{i,j}(\sigma;s)\bar{s}} t^{k(w)-1}\tilde{R}_{w\sigma s,\tau s}(t)\\ &= t\tilde{R}_{\sigma,\tau s}(t)+\tilde{R}_{\sigma s,\tau s}(t),\end{aligned}$$

by (21) and (22).

(iii) either $k\notin[i-1,j]$ or $\sigma(k)\notin[\sigma(i),\sigma(j)]$ or $\sigma(k+1)\notin[\sigma(i),\sigma(j)]$.

Then by Lemma 3.6 we have that $\mathscr{C}_{s(i),s(j)}(\sigma s)=\mathscr{C}_{i,j}(\sigma)$. Now, if $\sigma(k)>\sigma(k+1)$ (i.e., if $s\in D(\sigma)$) then it is easy to check that $s\in D(w\sigma)$ for all $w\in\mathscr{C}_{i,j}(\sigma)$ so that $\{w\in\mathscr{C}_{i,j}(\sigma): s\notin D(w\sigma)\}=\varnothing$ and hence, by (21) and (23),

$$\tilde{R}_{\sigma,\tau}(t)=\tilde{R}_{\sigma s,\tau s}(t).$$

On the other hand, if $\sigma(k)<\sigma(k+1)$ (i.e., if $s\notin D(\sigma)$) then it is easy to check that $s\notin D(w\sigma)$ for all $w\in\mathscr{C}_{i,j}(\sigma)$ so that $\{w\in\mathscr{C}_{i,j}(\sigma): s\notin D(w\sigma)\}=\mathscr{C}_{i,j}(\sigma)$. Furthermore, $s(i)=i$ (for if $k=i-1$ then $d=\tau(k+1)<\tau(k)=\sigma(k)<\sigma(k+1)\leqslant d$, and similarly if $k=i$ then $\sigma(i)<\sigma(k+1)$ and hence $d=\sigma(j)<\sigma(k+1)=\tau(k+1)<\tau(k)=d$). Therefore by (21), (22) and (23),

$$\tilde{R}_{\sigma,\tau}(t)=\tilde{R}_{\sigma s,\tau s}(t)+t\tilde{R}_{\sigma,\tau s}(t).$$

(iv) $k=i$, and $\sigma(i)<\sigma(i+1)\leqslant\sigma(j)$.

Then by Lemma 3.6 we have that $\mathscr{C}_{s(i),\, s(j)}(\sigma s) = \mathscr{C}_{i,\, j}(\sigma) \backslash \mathscr{C}_{i+1,\, j}(\sigma)\, \bar{s}$. Now, it is easy to see that, since $k = i$, $\{w \in \mathscr{C}_{i,\, j}(\sigma) : s \notin D(w\sigma)\} = \emptyset$. Therefore, we conclude from (23), and (21), that

$$\begin{aligned}
\tilde{R}_{\sigma,\, \tau}(t) &= \sum_{w \in \mathscr{C}_{s(i),\, s(j)}(\sigma s)} t^{k(w)-1} \tilde{R}_{w\sigma s,\, \tau s}(t) + \sum_{w \in \mathscr{C}_{i+1,\, j}(\sigma)\bar{s}} t^{k(w)-1} \tilde{R}_{w\sigma s,\, \tau s}(t) \\
&= \tilde{R}_{\sigma s,\, \tau s}(t) + \sum_{w \in \mathscr{C}_{i+1,\, j}(\sigma)} t^{k(w\bar{s})-1} \tilde{R}_{w\bar{s}\sigma s,\, \tau s}(t) \\
&= \tilde{R}_{\sigma s,\, \tau s}(t) + \sum_{w \in \mathscr{C}_{s(i),\, j}(\sigma)} t^{k(w)} \tilde{R}_{w\sigma,\, \tau s}(t) \\
&= \tilde{R}_{\sigma s,\, \tau s}(t) + t\tilde{R}_{\sigma,\, \tau s}(t),
\end{aligned}$$

by (22).

(v) $k = i-1$, and $\sigma(i) < \sigma(i-1) < \sigma(j)$.

This case can never happen because, since $s \in D(\tau)$ we have that $\tau(k) > \tau(k+1)$ and therefore $\tau(i-1) > \tau(i)$. But $\tau(i) = \tau(\tau^{-1}(d)) = d$. Hence $\tau(i-1) > d$, but, by the definition of $d \stackrel{\text{def}}{=} d(\tau, \sigma)$, this implies that $\tau(i-1) = \sigma(i-1)$. Hence $\sigma(i-1) > d = \sigma(j)$, which is a contradiction since we are assuming that $\sigma(i-1) < \sigma(j)$.

This concludes the proof. ∎

We can now state the precise relationship between the polynomials defined by (19) and the R-polynomials of S_n.

Corollary 3.8. *Let $\sigma, \tau \in S_n$, then*

$$R_{\sigma,\, \tau}(q) = q^{(l(\tau)-l(\sigma))/2} \tilde{R}_{\sigma,\, \tau}(q^{1/2} - q^{-1/2}).$$

Proof. This follows immediately from Theorems 2.2 and 3.7. ∎

As an immediate consequence of the preceding result, of Proposition 3.1, and of the definition (19) we obtain the following recurrence for R-polynomials.

Corollary 3.9. *Let $\sigma, \tau \in S_n$ be such that $\sigma < \tau$. Then*

$$R_{\sigma,\, \tau}(q) = \sum_{w \in \mathscr{C}_{\tau^{-1}(d),\, \sigma^{-1}(d)}(\sigma)} q^{n(w,\, \sigma)}(q-1)^{k(w)-1} R_{w\sigma,\, \tau}(q), \tag{25}$$

where $d \stackrel{\text{def}}{=} d(\sigma, \tau)$. ∎

The preceding recurrence has several advantages over the one given by (4), both of a theoretical as well as practical nature. The main one is that (25) does not "branch off" into two cases as (4) does. This allows one to use (25) repeatedly and thus explicitly solve the recurrence, as will be done in the next section. While this is theoretically possible also with (4), the details are much simpler using (25). The second one is that the recurrence (25) does not change the second permutation. This is extremely useful in induction arguments, as will be seen in the proofs of Theorems 4.4 and 4.6 in the next section. The third one is that the recurrence (25) is much faster from a computational point of view. Already computing $R_{1342,\,4321}(q)$ by hand, using (4), and comparing with the example computed before Proposition 3.5, should convince the reader of this. However, we have implemented both recursions (4) and (25) on a computer (using MAPLE) and have been able to verify this directly. For example, computing the R-polynomial of any pair of permutations in S_8 takes (running MAPLE V on a Sun SparcStation SLC) less than 65 seconds using (25) while it takes more than 5 minutes to compute $R_{12\cdots 8,\, 8\cdots 21}(q)$ using (4). These MAPLE programs (which will run also on older versions of MAPLE) are available from the author upon request.

4. APPLICATIONS

In this section we apply our main result to the explicit computation of R-polynomials and Kazhdan–Lusztig polynomials. We begin by "solving" the recurrence relation (25). Let $\sigma, \tau \in S_n$, $\sigma \leqslant \tau$. An *R-chain* from σ to τ is a chain $\sigma = \sigma_0 < \sigma_1 < \cdots < \sigma_r = \tau$ such that:

(i) $d(\sigma_i, \tau) < d(\sigma_{i-1}, \tau)$;

(ii) $(\sigma_i)(\sigma_{i-1})^{-1} \in \mathscr{C}(\sigma_{i-1})$;

for all $i = 1, ..., r$. We denote by $\mathscr{R}(\sigma, \tau)$ the set of all R-chains from σ to τ. Given any chain $C = (\sigma_0 < \sigma_1 < \cdots < \sigma_r)$ in S_n we define its *R-length* to be

$$l_R(C) \stackrel{\text{def}}{=} \sum_{i=1}^{r} (k((\sigma_i)(\sigma_{i-1})^{-1}) - 1).$$

For example, $C = (1234 < 4132 < 4312 < 4321)$ is an R-chain from 1234 to 4321 and its R-length is $l_R(C) = 2 + 1 + 1 = 4$.

THEOREM 4.1. *Let $\sigma, \tau \in S_n$, $\sigma \leqslant \tau$. Then*

$$R_{\sigma,\tau}(q) = \sum_{C \in \mathscr{R}(\sigma,\tau)} q^{(l(\tau) - l(\sigma) - l_R(C))/2} (q-1)^{l_R(C)}. \tag{26}$$

Proof. We will prove that

$$\tilde{R}_{\sigma,\tau}(t) = \sum_{C \in \mathscr{R}(\sigma,\tau)} t^{l_R(C)}, \tag{27}$$

and the thesis will then follow from Corollary 3.8. We proceed by induction on $d(\sigma, \tau)$, (27) being true by definition if $\sigma = \tau$. So let $\sigma, \tau \in S_n$ be such that $\sigma < \tau$. By (19) and our induction hypothesis we have that

$$\tilde{R}_{\sigma,\tau}(t) = \sum_{w \in \mathscr{C}_{i,j}(\sigma)} t^{k(w)-1} \sum_{C \in \mathscr{R}(w\sigma,\tau)} t^{l_R(C)}, \tag{28}$$

where $i \stackrel{\text{def}}{=} \tau^{-1}(d)$, $j \stackrel{\text{def}}{=} \sigma^{-1}(d)$, and $d \stackrel{\text{def}}{=} d(\sigma, \tau)$. Now, if $C = (\sigma_0 < \cdots < \sigma_r) \in \mathscr{R}(w\sigma, \tau)$ and $w \in \mathscr{C}_{i,j}(\sigma)$ then clearly $C_w \stackrel{\text{def}}{=} (\sigma < w\sigma < \sigma_1 < \cdots < \sigma_{r-1} < \tau) \in \mathscr{R}(\sigma, \tau)$ and

$$l_R(C_w) = k(w) - 1 + l_R(C).$$

Conversely, if $C' = (\sigma_0 < \sigma_1 < \cdots < \sigma_r) \in \mathscr{R}(\sigma, \tau)$ then it follows from our definitions that there exists $w \stackrel{\text{def}}{=} (\sigma(i_k), ..., \sigma(i_1)) \in \mathscr{C}(\sigma)$ (where $i_1 < \cdots < i_k$ and $\sigma(i_1) < \cdots < \sigma(i_k)$) such that $\sigma_1 = w\sigma$ and $C'' \stackrel{\text{def}}{=} (\sigma_1 < \cdots < \sigma_r) \in \mathscr{R}(w\sigma, \tau)$. Furthermore, since $d(\sigma_1, \tau) < d(\sigma_0, \tau) = d$, we have that

$$\sigma^{-1}(d) \neq \tau^{-1}(d) = (\sigma_1)^{-1}(d) = (w\sigma)^{-1}(d) = \sigma^{-1}(w^{-1}(d)), \tag{29}$$

and

$$\sigma^{-1}(h) = \tau^{-1}(h) = (\sigma_1)^{-1}(h) = (w\sigma)^{-1}(h) = \sigma^{-1}(w^{-1}(h))$$

if $h > d$. Hence $d \notin F(w)$ but $h \in F(w)$ if $h > d$, and this implies that $\sigma(i_k) = d$ and hence that $i_k = j$. Furthermore, it follows from this and (29) that $i = \tau^{-1}(d) = \sigma^{-1}(w^{-1}(d)) = \sigma^{-1}(w^{-1}(\sigma(i_k))) = \sigma^{-1}(\sigma(i_1)) = i_1$. Hence $w \in \mathscr{C}_{i,j}(\sigma)$ and it is clear that $l_R(C') = l_R(C'') + k - 1$. This shows that we have a bijection $\varphi\colon \bigcup_{w \in \mathscr{C}_{i,j}(\sigma)} \mathscr{R}(w\sigma, \tau) \to \mathscr{R}(\sigma, \tau)$ such that $l_R(\varphi(C)) = k(w) - 1 + l_R(C)$ for any $C \in \mathscr{R}(w\sigma, \tau)$ and $w \in \mathscr{C}_{i,j}(\sigma)$, and so (27) follows from (28), as desired. ∎

Note that, by Proposition 3.1 and our definitions, $l(\tau) - l(\sigma) - l_R(C)$ is even for any $\sigma, \tau \in S_n$ and $C \in \mathscr{R}(\sigma, \tau)$, so that (26) does not involve $q^{1/2}$. It would be interesting to find a direct connection between (26) and Deodhar's formula (see, e.g., [8], Theorem 1.3, or [16], p. 154). This should involve some bijection between R-chains and "distinguished subexpressions" (as defined in [8], Definition 2.3 or [16], p. 154). We should also note that a formula closely related to (27) was obtained (independently) by Dyer in [11].

We now concentrate on specific types of intervals in S_n for which the R-polynomial can be evaluated in closed form. Corollary 3.9 shows that the process of going from σ to $w\sigma$, where $w \in \mathscr{C}(\sigma)$, is the "basic step" in the computation of R-polynomials. This suggests that the R-polynomial of intervals of the form $[\sigma, w\sigma]$, where $w \in \mathscr{C}(\sigma)$, might be particularly simple. This is indeed the case (see Theorem 4.6) though the proof of this fact is by no means easy. We begin by establishing some preliminary facts about the Bruhat order of S_n which will be needed in the proof. Throughout the rest of this section we will use the characterization of Bruhat order given by Theorem 2.5 without explicit mention.

First, we need to define some refinements of the distance function on S_n which was introduced in § 3. Let $\sigma, \tau \in S_n$. For $i \in [n]$ let

$$d_i(\sigma, \tau) \stackrel{\text{def}}{=} \max\{j \in [n] : \sigma^{-1}(j) \neq \tau^{-1}(j), \sigma^{-1}(j) \in [i]\}, \tag{30}$$

(where $\max\{\varnothing\} \stackrel{\text{def}}{=} 0$). For example, if $\sigma = 198265374$ and $\tau = 298461357$ then $(d_1(\sigma, \tau), ..., d_9(\sigma, \tau)) = (1, 1, 1, 2, 2, 5, 5, 7, 7)$ and $(d_1(\tau, \sigma), ..., d_9(\tau, \sigma)) = (2, 2, 2, 4, 4, 4, 4, 5, 7)$. Note that $0 \leqslant d_1(\sigma, \tau) \leqslant d_2(\sigma, \tau) \leqslant \cdots \leqslant d_n(\sigma, \tau) = d(\sigma, \tau)$ and that, in general, $d_i(\sigma, \tau) \neq d_i(\tau, \sigma)$ unless $i = n$. In relation to Bruhat order the functions $d_i \colon S_n \times S_n \to \mathbf{N}$ have the following important property.

Proposition 4.2. *Let $\sigma, \tau \in S_n$ be such that $\sigma \leqslant \tau$. Then*

$$d_i(\sigma, \tau) \leqslant d_i(\tau, \sigma)$$

for $i = 1, ..., n$.

Proof. Fix $i \in [n]$. Let $\{a_1, ..., a_i\}_< \stackrel{\text{def}}{=} \{\sigma(1), ..., \sigma(i)\}$,

$$\{b_1, ..., b_i\}_< \stackrel{\text{def}}{=} \{\tau(1), ..., \tau(i)\}, \tag{31}$$

and $j, k \in [i]$ be such that $a_j = d_i(\sigma, \tau)$, $b_k = d_i(\tau, \sigma)$. Note that, since $\sigma \leqslant \tau$, we have that

$$a_r \leqslant b_r \tag{32}$$

for all $r = 1, ..., i$. In particular this implies that if $a_m = b_n$ for some $m, n \in [i]$ then $m \geqslant n$ (for if $m < n$ then $b_n = a_m < a_n \leqslant b_n$ by (32)). Furthermore, the definition of $d_i(\tau, \sigma)$ implies that

$$\sigma^{-1}(b_r) = \tau^{-1}(b_r) \tag{33}$$

for all $k+1 \leqslant r \leqslant i$. In particular this implies that $b_r \in \{\sigma(1), ..., \sigma(i)\}$, for all $k+1 \leqslant r \leqslant i$ (since $b_r = \sigma(\tau^{-1}(b_r))$ and $\tau^{-1}(b_r) \in [i]$ by (31)). But then $a_i = b_i$ (for if $b_i = a_r$ for some $r < i$, then $b_i = a_r < a_i \leqslant b_i$). Similarly $a_{i-1} = b_{i-1}$ (for if $b_{i-1} = a_r$ for some $r < i-1$, then $b_{i-1} = a_r < a_{i-1} \leqslant b_{i-1}$), and $a_{i-2} = b_{i-2}, ..., a_{k+1} = b_{k+1}$. But then $d_i(\sigma, \tau) \notin \{a_{k+1}, ..., a_i\}$ (otherwise $d_i(\sigma, \tau) \in \{b_{k+1}, ..., b_i\}$ and hence $\sigma^{-1}(d_i(\sigma, \tau)) = \tau^{-1}(d_i(\sigma, \tau))$ by (33), which contradicts the definition of $d_i(\sigma, \tau)$). Therefore $a_j \notin \{a_{k+1}, ..., a_i\}$, hence $j \leqslant k$ which implies that

$$d_i(\sigma, \tau) = a_j \leqslant a_k \leqslant b_k = d_i(\tau, \sigma),$$

as desired. ∎

Note that the converse of the above proposition does not hold. For example, if $\sigma = 312$ and $\tau = 231$ then $(d_1(\sigma, \tau), ..., d_3(\sigma, \tau)) = (3, 3, 3)$ and $(d_1(\tau, \sigma), ..., d_3(\tau, \sigma)) = (2, 3, 3)$.

We now need to prove some preliminary results about $\tilde{R}$ and R-polynomials.

Lemma 4.3. *Let $\sigma \in S_n$, $i, j \in [n]$, and $w \in \mathscr{C}_{i,j}(\sigma)$. Then there exists $v \in S_n$ such that*:

(i) $\tilde{R}_{\sigma, w\sigma}(t) = \tilde{R}_{\sigma v, w\sigma v}(t)$;

(ii) $w \in \mathscr{C}_{i,j}(\sigma v)$;

(iii) $m(w, \sigma v) = 0$.

Proof. We proceed by induction on $p(w, \sigma)$. If $p(w, \sigma) = 0$ then $m(w, \sigma) = 0$ and there is nothing to prove. So assume that $p(w, \sigma) \geqslant 1$ and let $k \stackrel{\text{def}}{=} k(w)$, for brevity, and

$$w \stackrel{\text{def}}{=} (\sigma(i_k), ..., \sigma(i_1)),$$

where $i = i_1 < \cdots < i_k = j$, and $\sigma(i_1) < \cdots < \sigma(i_k)$. Since $p(w, \sigma) \geqslant 1$ we have that $m(w, \sigma) \geqslant 1$ and hence there exists $2 \leqslant a \leqslant k-1$ such that $T_\sigma[i_{a-1}, i_a; \sigma(i_a), \sigma(j)] \neq \emptyset$. So let $b \in T_\sigma[i_{a-1}, i_a; \sigma(i_a), \sigma(j)]$, and $u \stackrel{\text{def}}{=} (b, b+1, b+2, ..., i_a)$, so that $u = s_b s_{b+1} s_{b+2} \cdots s_{i_a - 1}$. Since every element of $[n] \backslash \{\sigma(i_1), ..., \sigma(i_k)\}$ is fixed by w we have that $s_c \in D(\sigma s_b s_{b+1} \cdots s_{c-1})$ if and only if $s_c \in D(w\sigma s_b s_{b+1} \cdots s_{c-1})$, for all $b \leqslant c < i_a - 1$. Furthermore, since $\sigma(b) > \sigma(i_a) > \sigma(i_{a-1}) = (w\sigma)(i_a)$ we also have that $s_{i_a - 1} \in D(\sigma s_b s_{b+1} \cdots s_{i_a - 2}) \cap D(w\sigma s_b s_{b+1} \cdots s_{i_a - 2})$. Using Theorem 3.7 repeatedly we conclude from this that

$$\tilde{R}_{\sigma, w\sigma}(t) = \tilde{R}_{\sigma s_b s_{b+1} \cdots s_{i_a} - 1,\, w\sigma s_b s_{b+1} \cdots s_{i_a - 1}}(t) = \tilde{R}_{\sigma u, w\sigma u}(t).$$

Furthermore, it is clear that $w \in \mathscr{C}_{i,j}(\sigma w)$ (since $u(i)=i, u(j)=j$, and $w=(\sigma(i_k), ..., \sigma(i_1))=((\sigma u)(i_k), ..., (\sigma u)(i_a-1), ..., (\sigma u)(i_1)))$ and that $p(w, \sigma u) \leqslant p(w, \sigma)-1$. Hence, by our induction hypotheses, there exists $v \in S_n$ such that

$$\tilde{R}_{\sigma u,\, w\sigma u}(t)=\tilde{R}_{\sigma uv,\, w\sigma uv}(t),$$

$w \in \mathscr{C}_{i,j}(\sigma uv)$, and $m(w, \sigma uv)=0$, and the thesis follows. ∎

Our next preliminary result is also of independent interest. It gives sufficient conditions on two permutations so that their R-polynomial and Kazhdan–Lusztig polynomial factor. Let $\sigma \in S_n$, and $i, j \in [n], i \leqslant j$. We define the *restriction* of σ to $[i, j]$ to be the unique permutation $\sigma_{[i,j]} \in S([i, j])$ such that

$$\sigma^{-1}(\sigma_{[i,j]}(i))<\sigma^{-1}(\sigma_{[i,j]}(i+1))< \cdots <\sigma^{-1}(\sigma_{[i,j]}(j)).$$

For example, if $\sigma=7251634$ then $\sigma_{[3,5]}=534$ (i.e., $\sigma_{[3,5]}(3)=5$, $\sigma_{[3,5]}(4)=3$, $\sigma_{[3,5]}(5)=4$). Note that $\sigma_{[i,j]}=\mathrm{Id}([i, j])$ if and only if $\sigma^{-1}(i)<\sigma^{-1}(i+1)< \cdots <\sigma^{-1}(j)$, and that if $\sigma([i, j])=[i, j]$ then $\sigma_{[i,j]}=\sigma|_{[i,j]}$.

THEOREM 4.4. *Let $\sigma, \tau \in S_n, \sigma \leqslant \tau$. Suppose that there exist $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ such that $\sigma^{-1}((i_j, i_{j+1}])=\tau^{-1}((i_j, i_{j+1}])$ for all $j=0, ..., k$ (where $i_0 \overset{\text{def}}{=} 0$, $i_{k+1} \overset{\text{def}}{=} n$). Then*

$$R_{\sigma,\tau}(q)=\prod_{j=0}^{k} R_{\sigma_{(i_j, i_{j+1}]},\, \tau_{(i_j, i_{j+1}]}}(q), \tag{34}$$

and

$$P_{\sigma,\tau}(q)=\prod_{j=0}^{k} P_{\sigma_{(i_j, i_{j+1}]},\, \tau_{(i_j, i_{j+1}]}}(q). \tag{35}$$

Proof. We will prove first that

$$\tilde{R}_{\sigma,\tau}=\prod_{j=0}^{k} \tilde{R}_{\sigma_{(i_j, i_{j+1}]},\, \tau_{(i_j, i_{j+1}]}}(t), \tag{36}$$

and (34) will then follow from Corollary 3.8 and the easily established fact that

$$l(\tau)-l(\sigma)=\sum_{j=0}^{k} (l(\tau_{(i_j, i_{j+1}]})-l(\sigma_{(i_j, i_{j+1}]})), \tag{37}$$

in the hypotheses of the theorem. We proceed by induction on $d(\sigma, \tau)$, (36) being trivially true if $d(\sigma, \tau)=0$. So let σ, τ, and $i_0, ..., i_{k+1}$ be as in the statement of the theorem, with $\sigma<\tau$, and let, for brevity, $I_j \overset{\text{def}}{=} (i_j, i_{j+1}]$, for $j=0, ..., k$. By (19) we have that

$$\tilde{R}_{\sigma, \tau}(t)=\sum_{w \in \mathscr{C}_{i, j}(\sigma)} t^{k(w)-1} \tilde{R}_{w\sigma, \tau}(t) \tag{38}$$

where $i \overset{\text{def}}{=} \tau^{-1}(d)$, $j \overset{\text{def}}{=} \sigma^{-1}(d)$, and $d \overset{\text{def}}{=} d(\sigma, \tau)$. Let $r \in [k]$ be such that $d \in I_r$. Then $\tau(i) \in I_r$, and hence, by our hypotheses, $i \in \sigma^{-1}(I_r)$ so that $\sigma(i) \in I_r$. Therefore $\sigma(i), \sigma(j) \in I_r$ and this shows that, for any $w \in \mathscr{C}_{i, j}(\sigma)$, all elements of $[n]$ which are not fixed by w are in I_r. Therefore $w^{-1}(I_j)=I_j$ for all $j=0, ..., k$ and hence

$$(w\sigma)^{-1}(I_j)=\tau^{-1}(I_j)$$

for $j=0, ..., k$. By our induction hypotheses we then have that

$$\tilde{R}_{w\sigma, \tau}(t)=\prod_{j=0}^{k} \tilde{R}_{(w\sigma)_{I_j}, \tau_{I_j}}(t). \tag{39}$$

But, since every element of I_j, for $0 \leqslant j \leqslant k$, $j \neq r$, is fixed by w, we have that $(w\sigma)_{I_j}=\sigma_{I_j}$, for $j=0, ..., k$, $j \neq r$. Hence we conclude from (39) that

$$\tilde{R}_{w\sigma, \tau}(t)=\tilde{R}_{(w\sigma)_{I_r}, \tau_{I_r}}(t) \prod_{j \neq r} \tilde{R}_{\sigma_{I_j}, \tau_{I_j}}(t).$$

Substituting this in (38) we obtain that

$$\tilde{R}_{\sigma, \tau}(t)=\prod_{j \neq r} \tilde{R}_{\sigma_{I_j}, \tau_{I_j}}(t) \sum_{w \in \mathscr{C}_{i, j}(\sigma)} t^{k(w)-1} \tilde{R}_{(w\sigma)_{I_r}, \tau_{I_r}}(t). \tag{40}$$

Now it is clear that $d(\sigma_{I_r}, \tau_{I_r})=d(\sigma, \tau)-i_r$ (recall that we identify $S(I_r)$ with $S_{|I_r|}$), that restriction to I_r gives a bijection from $\mathscr{C}_{i, j}(\sigma)$ to $\mathscr{C}_{(\tau_{I_r})^{-1}(d-i_r), (\sigma_{I_r})^{-1}(d-i_r)}(\sigma_{I_r})$ and that, since w fixes all elements not in I_r, $(w\sigma)_{I_r}=w_{I_r}\sigma_{I_r}$. Hence

$$\sum_{w \in \mathscr{C}_{i, j}(\sigma)} t^{k(w)-1} \tilde{R}_{(w\sigma)_{I_r}, \tau_{I_r}}(t)=\sum_{w_{I_r} \in \mathscr{C}_{i', j'}(\sigma_{I_r})} t^{k(w_{I_r})-1} \tilde{R}_{w_{I_r}\sigma_{I_r}, \tau_{I_r}}(t)=\tilde{R}_{\sigma_{I_r}, \tau_{I_r}}(t),$$

by (19), where $i' \overset{\text{def}}{=} (\tau_{I_r})^{-1}(d-i_r)$ and $j' \overset{\text{def}}{=} (\sigma_{I_r})^{-1}(d-i_r)$, and (36) follows from (40).

We now prove (35) by induction on $l(\tau)-l(\sigma)$, (35) being clearly true if $\sigma=\tau$. Note first that it follows easily from our hypotheses, and

Theorem 2.5, that the map $a \mapsto (a_{I_0}, ..., a_{I_k})$ is a poset isomorphism from $[\sigma, \tau]$ to $[\sigma_{I_0}, \tau_{I_0}] \times \cdots \times [\sigma_{I_k}, \tau_{I_k}]$. Therefore we conclude from (34), (5), and our induction hypotheses that

$$\begin{aligned} q^{l(\tau)-l(\sigma)} P_{\sigma, \tau}\left(\frac{1}{q}\right) - P_{\sigma, \tau}(q) &= \sum_{\sigma < a \leqslant \tau} R_{\sigma, a}(q)\, P_{a, \tau}(q) \\ &= \prod_{j=0}^{k} \sum_{a_j \in [\sigma_{I_j}, \tau_{I_j}]} R_{\sigma_{I_j}, a_j}(q)\, P_{a_j, \tau_{I_j}}(q) - \prod_{j=0}^{k} P_{\sigma_{I_j}, \tau_{I_j}}(q) \\ &= \prod_{j=0}^{k} q^{l(\tau_{I_j}) - l(\sigma_{I_j})} P_{\sigma_{I_j}, \tau_{I_j}}\left(\frac{1}{q}\right) - \prod_{j=0}^{k} P_{\sigma_{I_j}, \tau_{I_j}}(q) \\ &= q^{l(\tau)-l(\sigma)} \prod_{j=0}^{k} P_{\sigma_{I_j}, \tau_{I_j}}\left(\frac{1}{q}\right) - \prod_{j=0}^{k} P_{\sigma_{I_j}, \tau_{I_j}}(q). \end{aligned} \tag{41}$$

Now, from (37) and part iii) of Theorem 2.3 we conclude that

$$\deg\left(\prod_{j=0}^{k} P_{\sigma_{I_j}, \tau_{I_j}}(q)\right) \leqslant \sum_{j=0}^{k} \left\lfloor \frac{l(\tau_{I_j}) - l(\sigma_{I_j}) - 1}{2} \right\rfloor \leqslant \left\lfloor \frac{l(\tau) - l(\sigma) - 1}{2} \right\rfloor.$$

Hence equating the coefficients of q^i, for $i = 0, ..., \lfloor (l(\tau) - l(\sigma) - 1)/2 \rfloor$, on both sides of (41) yields (35), as desired. ∎

We illustrate the preceding theorem with an example. Let $\sigma = 16573824$, and $\tau = 47583612$. Then $\sigma^{-1}([1, 4]) = \{1, 5, 7, 8\} = \tau^{-1}([1, 4])$ and $\sigma^{-1}([5, 8]) = \{2, 3, 4, 6\} = \tau^{-1}([5, 8])$, hence $R_{16573824,\, 47583612}(q) = R_{\sigma[1, 4],\, \tau[1, 4]}(q)\, R_{\sigma[5, 8],\, \tau[5, 8]}(q) = R_{1324,\, 4312}(q)\, R_{6578,\, 7586}(q) = R_{1324,\, 4312}(q) \times R_{2134,\, 3142}(q) = (q(q-1)^2 + (q-1)^4)(q-1)^2 = (q-1)^4\, (q^2 - q + 1)$, and $P_{16573824,\, 47583612}(q) = P_{\sigma[1, 4],\, \tau[1, 4]}(q)\, P_{\sigma[5, 8],\, \tau[5, 8]}(q) = P_{1324,\, 4312}(q)$ $P_{6578,\, 7586}(q) = P_{1324,\, 4312}(q)\, P_{2134,\, 3142}(q) = 1 \cdot 1 = 1$.

We note the following interesting consequence of Theorem 4.4.

COROLLARY 4.5. *The product of two R-polynomials (respectively, Kazhdan–Lusztig polynomials) is again an R-polynomial (respectively a Kazhdan–Lusztig polynomial).* ∎

For example, $R_{1324,\, 3412}(q)\, R_{123,\, 321}(q) = R_{1324567,\, 3412765}(q) = R_{5671324,\, 7653412}(q) = R_{5136274,\, 7346152}(q)$, $P_{1234,\, 3412}(q)\, P_{13425,\, 34512}(q) = P_{123457869,\, 341278956}(q) = P_{152378649,\, 374189526}(q) = P_{571823694,\, 783941562}(q)$, etc $\cdots$.

We now come to the second main result of this section.

THEOREM 4.6. *Let $\sigma \in S_n$ and $w \in \mathscr{C}(\sigma)$. Then*

$$R_{\sigma, w\sigma}(q) = (q-1)^{k(w)-1}(q^2-q+1)^{n(w,\sigma)}.$$

Proof. We will prove that

$$\widetilde{R}_{\sigma, w\sigma}(t) = t^{k(w)-1}(t^2+1)^{n(w,\sigma)}, \tag{42}$$

and the thesis will then follow from Corollary 3.8 and Proposition 3.1. We proceed by induction on $n(w,\sigma)+k(w)-1$, (42) being clearly true if $n(w,\sigma)+k(w)=2$ (i.e., if $n(w,\sigma)=0$ and $k(w)=2$). So let $i, j \in [n]$, $i<j$, be such that $w \in \mathscr{C}_{i,j}(\sigma)$ and let $k \stackrel{\text{def}}{=} k(w)$, for brevity, and

$$w \stackrel{\text{def}}{=} (\sigma(i_k), ..., \sigma(i_1)),$$

where $i = i_1 < \cdots < i_k = j$, and $\sigma(i_1) < \cdots < \sigma(i_k)$. By Lemma 4.3 we may assume that $m(w,\sigma)=0$, (i.e., that $T_\sigma[i_r, i_{r+1}; \sigma(i_s), \sigma(i_{s+1})] = \varnothing$ for all $1 \leqslant r < s \leqslant k-1$). So let

$$\{i_{r,1}, ..., i_{r,n_r}\}_< \stackrel{\text{def}}{=} T_\sigma[i_r, i_{r+1}; \sigma(i_r), \sigma(i_{r+1})],$$

for $r = 1, ..., k-1$. Note that this implies that

$$\sum_{r=1}^{k-1} n_r = n(w,\sigma). \tag{43}$$

Now, since every element of $[n] \backslash \{\sigma(i_1), ..., \sigma(i_k)\}$ is fixed by w, we have that $s_b \in D(\sigma s_{a_1} \cdots s_{a_p})$ if and only if $s_b \in D(w\sigma s_{a_1} \cdots s_{a_p})$ whenever $b, a_1, ..., a_p \in \bigcup_{j=1}^{k+1} (i_{j-1}, i_j - 1)$ (where $i_0 \stackrel{\text{def}}{=} 0$, $i_{k+1} \stackrel{\text{def}}{=} n+2$). Using Theorem 3.7 we therefore conclude that

$$\widetilde{R}_{\sigma, w\sigma}(t) = \widetilde{R}_{\sigma s_{a_1} \cdots s_{a_p},\, w\sigma s_{a_1} \cdots s_{a_p}}(t)$$

whenever $a_1, ..., a_p \in \bigcup_{j=1}^{k+1} (i_{j-1}, i_j - 1)$. This shows that we may assume without loss of generality that

$$\sigma(i_{r,1}) > \cdots > \sigma(i_{r,n_r}) \tag{44}$$

for all $1 \leqslant r \leqslant k-1$. Now, by (19), we have that

$$\widetilde{R}_{\sigma, w\sigma}(t) = \sum_{v \in \mathscr{C}_{i,j}(\sigma)} t^{k(v)-1} \widetilde{R}_{v\sigma, w\sigma}(t). \tag{45}$$

We now claim that if $v \in \mathscr{C}_{i,j}(\sigma)$ and $\sigma(i_a) \in F(v)$ for some $2 \leqslant a \leqslant k-1$ then $v\sigma \nleqslant w\sigma$ (and hence, in particular, $\widetilde{R}_{v\sigma, w\sigma}(t) = 0$). In fact, let $v \in \mathscr{C}_{i,j}(\sigma)$ and

$2 \leqslant a \leqslant k-1$ be such that $\sigma(i_a) \in F(v)$. Note that if $b \in [n]$ is such that $(v\sigma)^{-1}(b) \neq (w\sigma)^{-1}(b)$ then $i_1 \leqslant \sigma^{-1}(b) \leqslant i_a$ and $b \leqslant \sigma(i_k)$, and this, since $m(w, \sigma) = 0$, implies that $b \leqslant \sigma(i_a)$. Therefore, by (30),

$$d_{i_a}(v\sigma, w\sigma) \leqslant \sigma(i_a), \tag{46}$$

and $d_{i_a}(w\sigma, v\sigma) \leqslant \sigma(i_a)$. Now, if $\sigma(i_a) \in F(v)$ then $(v\sigma)^{-1}(\sigma(i_a)) = i_a$ and $(w\sigma)^{-1}(\sigma(i_a)) = \sigma^{-1}(w^{-1}(\sigma(i_a))) = \sigma^{-1}(\sigma(i_{a-1})) = i_{a-1}$ and this, by (30), shows that

$$d_{i_a}(v\sigma, w\sigma) = \sigma(i_a).$$

On the other hand, since $(w\sigma)^{-1}\,(\sigma(i_a)) > i_a$, it follows from (30) that $d_{i_a}(w\sigma, v\sigma) < \sigma(i_a)$. Hence

$$d_{i_a}(w\sigma, v\sigma) < d_{i_a}(v\sigma, w\sigma),$$

and this, by Proposition 4.2, implies that $v\sigma \nleqslant w\sigma$, as claimed.

It follows from the claim just proved and the inequalities (44) that if $v \in \mathscr{C}_{i,j}(\sigma)$ is such that $\tilde{R}_{v\sigma, w\sigma}(t) \neq 0$ then there exist $0 \leqslant s \leqslant k-1$, $1 \leqslant r_1 < r_2 < \cdots < r_s \leqslant k-1$, and $j_t \in [n_{r_t}]$, for $t = 1, ..., s$, such that

$$v = (\sigma(i_k), ..., \sigma(i_{r_s+1}), \sigma(i_{r_s, j_s}), \sigma(i_{r_s}), ..., \sigma(i_{r_1+1}), \sigma(i_{r_1, j_1}), \sigma(i_{r_1}), ..., \sigma(i_1)),$$

and conversely any such choice gives a $v \in \mathscr{C}_{i,j}(\sigma)$. In this case it is easy to see that

$$(v\sigma)^{-1}\,(\sigma(i_{r_t})) = i_{r_t, j_t} = (w\sigma)^{-1}\,(\sigma(i_{r_t, j_t})), \tag{47}$$

and

$$(v\sigma)^{-1}\,(\sigma(i_{r_t, j_t})) = i_{r_t+1} = (w\sigma)^{-1}\,(\sigma(i_{r_t})), \tag{48}$$

for $t = 1, ..., s$, while

$$(v\sigma)^{-1}\,(j) = (w\sigma)^{-1}\,(j) \tag{49}$$

if $j \notin \{\sigma(i_{r_1}), ..., \sigma(i_{r_s}), \sigma(i_{r_1, j_1}), ..., \sigma(i_{r_s, j_s})\}$. Therefore

$$(v\sigma)^{-1}\,((\sigma(i_{r_t, j_t}), \sigma(i_{r_{t+1}, j_{t+1}})]) = (w\sigma)^{-1}\,((\sigma(i_{r_t, j_t}), \sigma(i_{r_{t+1}, j_{t+1}})])$$

for $t = 0, ..., s$ (where $i_{r_0, j_0} \stackrel{\text{def}}{=} 0$, $i_{r_{s+1}, j_{s+1}} \stackrel{\text{def}}{=} n$). Hence we conclude from Theorem 4.4 (or, equivalently, from (36)) that

$$\tilde{R}_{v\sigma, w\sigma}(t) = \prod_{p=0}^{s} \tilde{R}_{(v\sigma)_{I_p}, (w\sigma)_{I_p}}(t) \tag{50}$$

where $I_p \stackrel{\text{def}}{=} (\sigma(i_{r_p, j_p}), \sigma(i_{r_{p+1}, j_{p+1}})]$, for $p = 0, ..., s$. But it follows from (47), (48), and (49) that $(w\sigma)_{I_{p-1}} = (\sigma(i_{r_p}), \sigma(i_{r_p, j_p}))(v\sigma)_{I_{p-1}}$ for $p = 1, ..., s$, and $(w\sigma)_{I_s} = (v\sigma)_{I_s}$. Since $n((\sigma(i_{r_p}), \sigma(i_{r_p, j_p})), (v\sigma)_{I_{p-1}}) = n_{r_p} - j_p < n(w, \sigma)$, for $p = 1, ..., s$, we conclude from our induction hypothesis that

$$\tilde{R}_{(v\sigma)_{I_{p-1}}, (w\sigma)_{I_{p-1}}}(t) = t(t^2+1)^{n_{r_p} - j_p} \tag{51}$$

for $p = 1, ..., s$. Hence we obtain from (50) and (51) that

$$\tilde{R}_{v\sigma, w\sigma}(t) = \prod_{p=1}^{s} t(t^2+1)^{n_{r_p} - j_p}.$$

Therefore we conclude from (45) that

$$\begin{aligned}
\tilde{R}_{\sigma, w\sigma}(t) &= \sum_{s=0}^{k-1} \sum_{1 \leqslant r_1 < \cdots < r_s \leqslant k-1} \sum_{j_1=1}^{n_{r_1}} \cdots \sum_{j_s=1}^{n_{r_s}} t^{k+s-1} \prod_{p=1}^{s} t(t^2+1)^{n_{r_p} - j_p} \\
&= \sum_{s=0}^{k-1} \sum_{1 \leqslant r_1 < \cdots < r_s \leqslant k-1} t^{k+s-1} \prod_{p=1}^{s} \sum_{j_p=1}^{n_{r_p}} t(t^2+1)^{n_{r_p} - j_p} \\
&= \sum_{s=0}^{k-1} \sum_{1 \leqslant r_1 < \cdots < r_s \leqslant k-1} t^{k+s-1} \prod_{p=1}^{s} t\,\frac{(t^2+1)^{n_{r_p}} - 1}{(t^2+1) - 1} \\
&= \sum_{s=0}^{k-1} \sum_{1 \leqslant r_1 < \cdots < r_s \leqslant k-1} t^{k-1} \prod_{p=1}^{s} ((t^2+1)^{n_{r_p}} - 1) \\
&= t^{k-1} \sum_{r=1}^{k-1} (1 + ((t^2+1)^{n_r} - 1)) \\
&= t^{k-1} (t^2+1)^{n(w, \sigma)},
\end{aligned}$$

by (43), as desired. ∎

Theorems 4.1 and 4.6 enable us to characterize the R-polynomials of pairs of permutations that differ by a transposition.

Corollary 4.7. *Let $\sigma, \tau \in S_n$, $\sigma \leqslant \tau$. Then the following are equivalent:*

(i) *$R_{\sigma, \tau}(q)$ has a simple zero at $q = 1$;*

(ii) $\tau\sigma^{-1} \in T$;

(iii) *$R_{\sigma, \tau}(q) = (q-1)(q^2 - q + 1)^r$ for some $r \in \mathbf{N}$.*

Proof. It is clear that (iii) implies (i) and that (by Theorem 4.6) (ii) implies (iii). Now, if (i) holds then $[t](\tilde{R}_{\sigma,\tau}(t)) \neq 0$ (else $t^2 | \tilde{R}_{\sigma,\tau}(t)$ and hence $(q-1)^2 | R_{\sigma,\tau}(q)$ which contradicts (i)). By Theorem 4.1 (or, equivalently, by (27)) this implies that there is at least one R-chain from σ to τ of R-length one. Hence this R-chain has length one and therefore $\tau\sigma^{-1} \in \mathscr{C}(\sigma)$ and $k(\tau\sigma^{-1}) = 2$, which implies (ii), as desired. ▮

We now come to the third main result of this section. This gives a simple poset-theoretic condition on the interval $[\sigma, \tau]$ which insures that the corresponding R-polynomial is a power of $(q-1)$. Again, we need first a preliminary result on the Bruhat order of S_n.

Lemma 4.8. *Let $\sigma \in S_n$, and $\bar{s} \in T$ be such that $l(\bar{s}\sigma) - l(\sigma) > 1$. Then $[\sigma, \bar{s}\sigma]$ contains an interval isomorphic to S_3.*

Proof. Let $i, j \in [n]$, $i < j$, be such that $\bar{s} = (\sigma(i), \sigma(j))$. Since $l(\bar{s}\sigma) > l(\sigma) + 1$ this implies that $\sigma(i) < \sigma(j)$ and that $T_\sigma[i, j; \sigma(i), \sigma(j)] \neq \emptyset$. So let

$$\{k_1, ..., k_r\}_< \stackrel{\text{def}}{=} T_\sigma[i, j; \sigma(i), \sigma(j)].$$

We now proceed by induction on $l(\bar{s}\sigma) - l(\sigma)$. If $l(\bar{s}\sigma) - l(\sigma) = 3$ (note that $l(\bar{s}\sigma) - l(\sigma)$ is always odd) then $r = 1$ and it is easy to see that $[\sigma, \bar{s}\sigma] \cong S_3$ (as posets). So assume that $l(\bar{s}\sigma) - l(\sigma) \geqslant 5$, and let $\tau \stackrel{\text{def}}{=} (\sigma(k_r), \sigma(j))\sigma$. Then it is easy to check that

$$\sigma < \tau < \bar{s}\tau < \bar{s}\sigma, \tag{52}$$

and that $l(\bar{s}\tau) = l(\bar{s}\sigma) - 1$, $l(\tau) = l(\sigma) + 1$. Therefore

$$1 < l(\bar{s}\tau) - l(\tau) < l(\bar{s}\sigma) - l(\sigma).$$

Hence, by the induction hypothesis, $[\tau, \bar{s}\tau]$ contains an interval isomorphic to S_3 and hence (by (52)) $[\sigma, \bar{s}\sigma]$ also does, as desired. ▮

Theorem 4.9. *Let $\sigma, \tau \in S_n$, $\sigma \leqslant \tau$, be such that $[\sigma, \tau]$ does not contain an interval isomorphic to S_3. Then*

$$R_{\sigma,\tau}(q) = (q-1)^{l(\tau) - l(\sigma)}. \tag{53}$$

Proof. We proceed by induction on $d(\sigma, \tau)$, (53) being clearly true if $d(\sigma, \tau) = 0$. So assume that $\sigma < \tau$ and let $w \in \mathscr{C}_{i,j}(\sigma)$ be such that $w\sigma \leqslant \tau$, (where $i \stackrel{\text{def}}{=} \tau^{-1}(d)$, $j \stackrel{\text{def}}{=} \sigma^{-1}(d)$, $d \stackrel{\text{def}}{=} d(\sigma, \tau)$). Then

$$w = (\sigma(i_k), \sigma(i_{k-1}), ..., \sigma(i_1))$$

where $i = i_1 < i_2 < \cdots < i_k = j$, and $\sigma(i) = \sigma(i_1) < \sigma(i_2) < \cdots < \sigma(i_k) = \sigma(j)$. We now claim that $n(w, \sigma) = 0$. In fact, if $n(w, \sigma) \geqslant 1$, then, by definition, there is $r \in [k-1]$ and $i_r < a < i_{r+1}$ such that

$$\sigma(i_r) < \sigma(a) < \sigma(j).$$

But $w\sigma = \bar{s}_1 \cdots \bar{s}_{k-1}\sigma$, where $\bar{s}_l \overset{\text{def}}{=} (\sigma(i_l), \sigma(i_k))$, for $l = 1, ..., k-1$, hence

$$v \overset{\text{def}}{=} \bar{s}_{r+1} \cdots \bar{s}_{k-1}\sigma$$

is such that $l(\bar{s}_r v) - l(v) > 1$ and

$$\sigma \leqslant v < \bar{s}_r v \leqslant w\sigma \leqslant \tau. \tag{54}$$

Therefore, by Lemma 4.8, $[v, \bar{s}_r v]$ contains an interval isomorphic to S_3 and hence, by (54), $[\sigma, \tau]$ also does, which contradicts our hypothesis.

So $n(w, \sigma) = 0$ for any $w \in \mathscr{C}_{i,j}(\sigma)$ such that $w\sigma \leqslant \tau$. Since, by Proposition 3.2, there is a unique $w_0 \in \mathscr{C}_{i,j}(\sigma)$ such that $n(w_0, \sigma) = 0$, we conclude that there is at most one $w \in \mathscr{C}_{i,j}(\sigma)$ such that $w\sigma \leqslant \tau$, namely $w = w_0$. Therefore, from (25) we have that

$$R_{\sigma,\tau}(q) = (q-1)^{k(w_0)-1} R_{w_0\sigma,\tau}(q). \tag{55}$$

Since $\sigma \leqslant \tau$ we know that $R_{\sigma,\tau}(q) \neq 0$, hence, by (55), $R_{w_0\sigma,\tau}(q) \neq 0$ and therefore $w_0\sigma \leqslant \tau$. Hence $[w_0\sigma, \tau] \subsetneqq [\sigma, \tau]$ and therefore $[w_0\sigma, \tau]$ does not contain any interval isomorphic to S_3. Hence, by our induction hypotheses,

$$R_{w_0\sigma,\tau}(q) = (q-1)^{l(\tau) - l(w_0\sigma)}$$

and this, by (55) and Proposition 3.1, implies (53), as desired. ∎

As an immediate consequence of the preceding theorem we obtain the following result, which was first observed by M. Haiman and G. Kalai (private communication).

Corollary 4.10. *Let $\sigma, \tau \in S_n$, $\sigma \leqslant \tau$, be such that $[\sigma, \tau]$ is a lattice. Then*

$$R_{\sigma,\tau}(q) = (q-1)^{l(\tau) - l(\sigma)}. \quad ∎$$

Theorem 4.9 also has interesting and nontrivial consequences for Kazhdan–Lusztig polynomials. In fact, it implies the following result.

THEOREM 4.11. *Let $u, v \in S_n$, $u \leqslant v$, be such that $[u, v]$ does not contain an interval isomorphic to S_3. Then*

$$P_{u,v}(q) = g([u, v]^*; q). \tag{56}$$

In particular,

$$P_{u,v}(q) = 1 + \sum_{i=1}^{\lfloor d/2 \rfloor} (h_i - h_{i-1})\, q^i$$

where $d \overset{\text{def}}{=} l(v) - l(u) - 1$, and $(h_0, h_1, ..., h_d)$ is the h-vector of $[u, v]^*$.

Proof. We proceed by induction on $l(v) - l(u)$, the thesis being clearly true if $u = v$. Note that by Theorem 4.9 and our hypotheses we have that $R_{u,x}(q) = (q-1)^{l(x)-l(u)}$ for all $x \in [u, v]$. Therefore, using part (iv) of Theorem 2.3 and our induction hypothesis we obtain that

$$\begin{aligned} q^{d+1} P_{u,v}\left(\frac{1}{q}\right) - P_{u,v}(q) &= \sum_{u < x \leqslant v} R_{u,x}(q)\, P_{x,v}(q) \\ &= \sum_{u < x \leqslant v} (q-1)^{l(x)-l(u)}\, g([x, v]^*, q) \\ &= (q-1)\, f([u, v]^*; q), \end{aligned} \tag{57}$$

by (2). Since, by part (iii) of Theorem 2.3, $\deg(P_{u,v}(q)) \leqslant \lfloor d/2 \rfloor$, equating the coefficients of q^i, for $i = 0, ..., \lfloor d/2 \rfloor$, on both sides of (57) yields, by (1), that

$$P_{u,v}(q) = g([u, v]^*; q),$$

as desired. The second statement then follows from (56) and the well known fact (see, e.g., [25], Theorem 3.14.9, or [26], Theorem 2.4) that the h-vector of an Eulerian poset is symmetric, so that $h_i = h_{d-i}$ for $i = 0, ..., d$. ∎

It had long been noticed (see; e.g., [5], § 5.3, and [26], p. 200) that the g-polynomials of Eulerian posets have some striking similarities with Kazhdan–Lusztig polynomials. The preceding result is the first to establish a direct, precise connection, between the two. Note that equation (56) is in general false. For example, $P_{123,321}(q) = 1$ but $g([123, 321]^*; q) = 1 - q$.

We can now use some standard results from the theory of g-polynomials to compute the Kazhdan–Lusztig polynomials of some classes of intervals. For $n \in \mathbf{N}$ we denote by B_n and C_n the Boolean algebra of rank n and the face lattice of an n-dimensional cube, respectively.

COROLLARY 4.12. *Let $u, v \in S_n$, $u \leqslant v$, be such that $[u, v] \cong B_{l(v)-l(u)}$ (as posets). Then*

$$P_{u,v}(q) = 1.$$

Proof. This follows immediately from Theorem 4.11 and the well known fact that $g(B_d; q) = 1$ for all $d \in \mathbf{N}$ (see, e.g., [26], Proposition 2.1, or [25], Example 3.14.8). ▌

COROLLARY 4.13. *Let $u, v \in S_n$, $u \leqslant v$, be such that $[u, v] \cong C_d^*$ (as posets) where $d \stackrel{\text{def}}{=} l(v) - l(u) - 1$. Then*

$$P_{u,v}(q) = \sum_{i=0}^{\lfloor d/2 \rfloor} \frac{1}{d-i+1} \binom{d}{i} \binom{2(d-i)}{d} (q-1)^i. \tag{58}$$

Equivalently, $[q^i](P_{u,v}(q))$ is the number of plane trees with $d+1$ vertices such that exactly i vertices have $\geqslant 2$ sons, for all $i \in \mathbf{N}$. In particular, $\deg(P_{u,v}(q)) = \lfloor d/2 \rfloor$.

Proof. (58) follows immediately from Theorem 4.11 and Proposition 2.6 of [26] (see also [25], Chapter 3, Exercise 71(f)). The second statement was first proved by L. Shapiro (unpublished, see also [25], Chapter 3, Exercise 71(g)). ▌

One other consequence of Theorem 4.11 is the following.

PROPOSITION 4.14. *Let $u, v \in S_n$, $u \leqslant v$, be such that $[u, v]$ does not contain an interval isomorphic to S_3. Then*

$$[q](P_{u,v}(q)) = r_1 - d - 1,$$

and

$$[q^2](P_{u,v}(q)) = \binom{d+1}{2} - d r_1 + r_2 + r_3^c,$$

where $d \stackrel{\text{def}}{=} l(v) - l(u) - 1$, $r_i \stackrel{\text{def}}{=} |\{x \in [u, v] : l(v) - l(x) = i\}|$, for $i = 0, ..., d+1$, and $r_3^c \stackrel{\text{def}}{=} |\{x \in [u, v] : [x, v] \cong C_2\}|$.

Proof. Note first that it follows from Proposition 1 and Theorem 2 of [3], and from results of Jantzen (see [17], p. 177) that every interval in S_n

of length 2 or 3 is isomorphic to either B_2, B_3, S_3, or C_2. Since $[u, v]^*$ does not contain any interval isomorphic to S_3 we obtain from (2) that

$$f([u, v]^*; q) = (q-1)^d + r_1(q-1)^{d-1} g(B_1; q) + r_2(q-1)^{d-2} g(B_2; q) \\ + (q-1)^{d-3} (r_3^c g(C_2; q) + (r_3 - r_3^c)\, g(B_3; q)) + R(q) \quad (59)$$

where $R(q)$ is a polynomial in q of degree $\leqslant d-3$. But it follows easily from the definitions that $g(B_1; q) = g(B_2; q) = g(B_3; q) = 1$ and $g(C_2; q) = 1 + q$, hence equating the coefficients of q^{d-i}, for $i = 0, 1, 2$, on both sides of (59), and using (1) and Theorem 4.11 yields the result. ▮

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